Relative Gröbner and Involutive Bases for Ideals in Quotient Rings

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Notations

- $\mathcal{P} = \mathbb{K}[x_1, \ldots, x_n]$: polynomial ring
- $\prec$: fixed monomial ordering
- $\mathcal{M}$: set of monomials $\mathbf{x}^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$
- $\mathrm{lm}(f)$: leading monomial of $0 \neq f \in \mathcal{P}$
- $\mathrm{lm}(\mathcal{I}) = \langle \mathrm{lm}(f) \mid f \in \mathcal{I} \rangle$: leading ideal of $0 \neq \mathcal{I} \trianglelefteq \mathcal{P}$
- $\mathrm{Syz}(F) \subset \mathcal{P}^k$: syzygy module of $F = \{f_1, \ldots, f_k\} \subset \mathcal{P}$
- $[f] = [f]_{\mathcal{I}} = f + \mathcal{I}$: element of quotient ring $\mathcal{P}/\mathcal{I}$
- $\mathrm{NF}_G(f)$: normal form of $f \in \mathcal{P}$ w.r.t. a $\prec$-Gröbner basis G

Gröbner Bases

Definition

$G = \{g_1, \ldots, g_m\} \subset \mathcal{I}$ Gröbner basis of ideal $\mathcal{I} \trianglelefteq \mathcal{P}$, if $\langle \operatorname{lm}(g_1), \ldots, \operatorname{lm}(g_m) \rangle = \operatorname{lm}(\mathcal{I})$.

Definition

Let $f, g \in \mathcal{P}$ be two monic polynomials. Their S-Polynomial is $S(f, g) = (\operatorname{lcm}(\operatorname{lm}(f), \operatorname{lm}(g)) / \operatorname{lm}(f)) \cdot f - (\operatorname{lcm}(\operatorname{lm}(f), \operatorname{lm}(g)) / \operatorname{lm}(g)) \cdot g$.

Theorem (Buchberger 1965)

One can compute a Gröbner basis of any given ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ by producing new leading monomials via S-Polynomials.

- If $G = \{g_1, \ldots, g_m\}$ is a Gröbner basis, then $S(g_i, g_j) \rightarrow^*_G 0$. (S-Polynomial reduces to zero). $\rightsquigarrow$ syzygy $\mathbf{S}_{ij} \in \mathrm{Syz}(G)$.
- The free module $\mathcal{P}^m$ has module monomials $\mathbf{x}^\mu \mathbf{e}_k$, where $\mathbf{x}^\mu \in \mathcal{M}$ and $1 \leq k \leq m$.

Theorem (Schreyer 1980)

If $G = \{g_1, \ldots, g_m\}$ is a Gröbner basis, then the set $\{\mathbf{S}_{ij} \mid 1 \leq i < j \leq m\} \subset \mathrm{Syz}(G)$ is a Gröbner basis of $\mathrm{Syz}(G)$ with respect to a suitable ordering on module monomials.

Involutive Divisions

- Involutive division L: Restricted division relation for monomials.
- Given set $H \subset \mathcal{M}$ and $\mathbf{x}^\mu \in H$, assigns multiplicative variables $L(\mathbf{x}^\mu, H) \subseteq \{x_1, \ldots, x_n\}$.
- The involutive cones $\mathcal{C}_L(\mathbf{x}^\mu, H) = \{\mathbf{x}^\nu \mid \mathbf{x}^\nu / \mathbf{x}^\mu \in \mathbb{K}[L(\mathbf{x}^\mu, H)]\}$ intersect only trivially as $\mathbf{x}^\mu$ varies in H.
- Filter axiom: If the ambient set H of $\mathbf{x}^\mu$ shrinks to H', then $L(\mathbf{x}^\mu, H') \supseteq L(\mathbf{x}^\mu, H)$.

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Examples

1. Pommaret division P: monomial $\mathbf{x}^\mu$ with $\mu = (0, \ldots, 0, \mu_k, \ldots, \mu_n)$ $\rightsquigarrow P(\mathbf{x}^\mu) = \{x_1, \ldots, x_k\}$. (*global division*)
2. Janet division J (non-global): for instance, if $n = 3$, $H = \{x_1x_2x_3, x_2^2, x_3^2\}$, then $J(x_1x_2x_3, H) = \{x_1, x_2\}$, because $\deg_{x_3}(x_3^2) > \deg_{x_3}(x_1x_2x_3)$. But $J(x_1x_2x_3, H \setminus \{x_3^2\}) = \{x_1, x_2, x_3\}$.

Involutive Bases and Their Syzygies

- Finite subset $H \subset \mathcal{I} \trianglelefteq \mathcal{P}$ with distinct leading monomials is *strong L-involutive basis* of $\mathcal{I}$, if $\bigsqcup_{\mathbf{x}^\mu \in \operatorname{lm}(H)} \mathcal{C}_L(\mathbf{x}^\mu, \operatorname{lm}(H)) = \operatorname{lm}(\mathcal{I})$.
- $\rightsquigarrow$ direct sum decomposition $\mathcal{I} = \bigoplus_{h \in H} \mathbb{K}[L(lm(h), \operatorname{lm}(H))] \cdot h$
- Let $F \subset \mathcal{P}$ and $f \in F$. Non-multiplicative prolongation of f: product $x_\ell \cdot f$, where $x_\ell \notin L(\operatorname{lm}(f), \operatorname{lm}(F))$. $\rightsquigarrow$ involutive analogon of S-polynomials.
- Use *involutive* reductions to obtain syzygies

Theorem

Given a strong Pommaret (resp. Janet) basis H of an ideal $\mathcal{I} \trianglelefteq \mathcal{P}$, the set of all non-multiplicative prolongations induces a Pommaret (resp. Janet) basis of $\operatorname{Syz}(H)$ *for a suitably chosen order.*

Relative Gröbner Bases

- Ideal $\hat{\mathcal{J}} \trianglelefteq \mathcal{P}/\mathcal{I} \rightsquigarrow \hat{\mathcal{J}} = \mathcal{J}/\mathcal{I}$, where $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{P}$.
- $[h] \in \mathcal{P}/\mathcal{I}$, G Gröbner basis of $\mathcal{I} \rightsquigarrow$ define $\mathrm{lm}([h]) := \mathrm{lm}(NF_G(h))$.

Definition

1. $H = \{h_1, \ldots, h_k\} \subset \mathcal{J}$ Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$ if
$$\langle \mathrm{lm}(H)\rangle + \mathrm{lm}(\mathcal{I}) = \mathrm{lm}(\mathcal{J}).$$
2. $\hat{H} = \{[h_1], \ldots, [h_k]\} \subset \hat{\mathcal{J}}$ Gröbner basis, if $\langle \mathrm{lm}(\hat{H})\rangle + \mathrm{lm}(\mathcal{I}) = \mathrm{lm}(\mathcal{J})$.

Proposition

H Gröbner basis of $\mathcal{J} \implies$
$\{\mathrm{NF}_G(h) \mid h \in H \setminus \mathcal{I}\}$ Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$.

S-polynomials and A-polynomials

- $\mathcal{I} \trianglelefteq \mathcal{P}$, $F \subset \mathcal{P} \setminus \mathcal{I}$, $h \in \mathcal{P}$: Division of h by F relative to $\mathcal{I}$.
- Prioritize reduction modulo $\mathcal{I}$
- Can produce new leading monomials by:
  1. S-polynomials among the elements of F
  2. annihilating leading monomials modulo $\mathrm{lm}(\mathcal{I}) \rightsquigarrow$ A-polynomial
- G Gröbner basis of $\mathcal{I} \rightsquigarrow$ A-polynomials come from S-polynomials between F and G
- Relative Buchberger algorithm: Prioritize A-polynomials

Relative Schreyer Theorem

- $H \subset \mathcal{P} \setminus \mathcal{I}$ relative Gröbner basis, G Gröbner basis of $\mathcal{I}$
  $\rightsquigarrow$ Gröbner basis $H \cup G$ of $\mathcal{J} := \langle H, G \rangle$.
- S-polynomials and A-polynomials $\rightsquigarrow$ S-syzygies and A-syzygies

Theorem

H Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$

$$\Downarrow$$

$\{S\text{–}syzygies\} \cup \{A\text{–}syzygies\}$ *Gröbner basis of* $\mathrm{Syz}_{\mathcal{P}/\mathcal{I}}([h] \mid h \in H)$

Relative Involutive Divisions and Bases

Definition

$\mathcal{I}$ monomial ideal. Involutive division L on $\mathcal{M} \setminus \mathcal{I}$ relative to $\mathcal{I}$:
For $\mathbf{x}^\mu \in H \subset \mathcal{M} \setminus \mathcal{I}$: Multiplicative variables $L(\mathbf{x}^\mu, H) \rightsquigarrow$
Relative involutive cones $\mathcal{C}_{L,\mathcal{I}}(\mathbf{x}^\mu, H) = \mathcal{C}_L(\mathbf{x}^\mu, H) \setminus \mathcal{I}$.

Proposition

*L classical involutive division, $\mathcal{I}$ monomial ideal, $\mathbf{x}^\mu \in H$, $H \cap \mathcal{I} = \emptyset$.
Relative involutive division $L_\mathcal{I}$ induced by L:*

$$x_i \in L_\mathcal{I}(\mathbf{x}^\mu, H) \iff x_i \in L(\mathbf{x}^\mu, H) \ \vee \ x_i \cdot \mathbf{x}^\mu \in \mathcal{I}.$$

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Definition

$\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{P}$ and L relative involutive division on $\mathcal{M} \setminus \operatorname{Im}(\mathcal{I})$. Then:
$H = \{h_1, \dots, h_k\} \subset \mathcal{J} \setminus \mathcal{I}$ strong L-involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$, if

$$\bigsqcup_{h \in H} \mathcal{C}_{L,\mathcal{I}}(\operatorname{Im}(h), \operatorname{Im}(H)) = \operatorname{Im}(\mathcal{J}) \setminus \operatorname{Im}(\mathcal{I}).$$

Existence of Relative Involutive Bases

Definition

Involutive division L is *noetherian* if each monomial ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ has finite L-involutive basis.

Proposition

L classical noetherian division, $\mathcal{I}$ monomial ideal $\Longrightarrow$ relative division $L_{\mathcal{I}}$ also noetherian

- For *constructive* noetherian involutive divisions L, can compute $L_{\mathrm{lm}(\mathcal{I})}$-involutive bases of ideals in $\mathcal{P}/\mathcal{I}$
- In particular, algorithm for computing relative Janet bases
- Use ordinary Gröbner reductions for A-polynomials and relative involutive reductions for non-multiplicative prolongations

Minimal Relative Involutive Bases

Definition

1. Let $\mathcal{I} \subseteq \mathcal{J}$ be monomial ideals, $L_{\mathcal{I}}$ relative involutive division. Relative $L_{\mathcal{I}}$-involutive basis H of $\mathcal{J}$ is *minimal*, if it is a subset of any other such basis.
2. Let $\mathcal{I} \subseteq \mathcal{J}$ be polynomial ideals, $L_{\mathrm{lm}(\mathcal{I})}$ relative involutive division. Relative $L_{\mathrm{lm}(\mathcal{I})}$-involutive basis H of $\mathcal{J}$ is *minimal*, if $\mathrm{lm}(\bullet)$ is bijection from H to the minimal relative involutive basis of $\mathrm{lm}(\mathcal{J})$.

- Algorithm for computing minimal relative Janet bases
- If a relative Pommaret basis exists, can compute minimal such basis.

Relative Quasi-Stability and Quasi-Stable Position

- For an ideal $\mathcal{I}$, a strong Pommaret basis exists $\iff$ $\mathcal{I}$ is in *quasi-stable position*: combinatorial condition on $\mathrm{lm}(\mathcal{I})$.
- Analogously, strong Pommaret basis for $\mathcal{J}$ relative to $\mathcal{I}$ exists $\iff$ $\mathrm{lm}(\mathcal{J})$ is quasi-stable relative to $\mathrm{lm}(\mathcal{I})$:

Definition

Let $\mathcal{I} \subseteq \mathcal{J}$ be two monomial ideals. $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$ if for all $\mathbf{x}^\mu \in \mathcal{J} \setminus \mathcal{I}$ and $x_i \notin P(\mathbf{x}^\mu)$ there exists an $s \geq 0$ such that either $x_i^s \cdot \mathbf{x}^\mu \in \mathcal{I}$ or $x_i^s \cdot \mathbf{x}^\mu / x_{\mathrm{cls}(x^\mu)} \in \mathcal{J}$.

Proposition

Let $\mathcal{I} \subseteq \mathcal{J}$ be two monomial ideals. Then $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$ $\iff$ $\mathcal{J}$ has a finite Pommaret basis relative to $\mathcal{I}$.

Transforming to Quasi-Stable Position

- Special setting: $\mathcal{I} \subseteq \mathcal{J}$ *homogeneous* ideals, $\prec$ degree reverse lexicographical ordering with $x_1 \prec x_2 \prec \cdots \prec x_n$.
- One can transform $\mathcal{J}$ to quasi-stable position by concatenating linear coordinate transformations of the form $x_i \mapsto x_i + x_j$, where $i < j$ and all other variables are left unchanged.
- An algorithm for choosing indices i and j is induced by the following:

Proposition

Let $B \subset \mathrm{lm}(\mathcal{J}) \setminus \mathrm{lm}(\mathcal{I})$ be a Pommaret autoreduced set of monomials with $\langle B \rangle + \mathrm{lm}(\mathcal{I}) = \mathrm{lm}(\mathcal{J})$ for which Pommaret and Janet multiplicative variables relative to $\mathrm{lm}(\mathcal{I})$ coincide. Then $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$.

Sufficiently Generic Position

- Given $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{P}$ homogeneous, both quasi-stable position of $\mathcal{I}$ and relative quasi-stable position of $\mathcal{J}$ are generic positions
- Can find sparse linear coordinate transformation such that in new coordinates, $\mathcal{I}$ has finite minimal Pommaret basis G and $\mathcal{J}$ has finite minimal relative Pommaret basis H.

Involutive Syzygies: New Phenomena in Relative Setting

H classical or relative involutive basis, $h \in H$.
Non-multiplicative prolongation: $x_i \cdot h$ where $x_i \notin L(\mathrm{lm}(h), \mathrm{lm}(H))$.

Fact (Involutive Schreyer Theorem)

H strong Pommaret (resp. Janet) basis of $\mathcal{I} \Longrightarrow$ set of non-multiplicative prolongations induces Pommaret (resp. Janet) basis of $\mathrm{Syz}(H)$.

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Theorem

H strong Pommaret basis of $\mathcal{J}$ relative to $\mathcal{I}$, G strong Pommaret basis of $\mathcal{I} \Longrightarrow$ syzygies from non-multiplicative prolongations of H together with A-syzygies of $H \cup G$ form relative Pommaret basis of $\mathrm{Syz}(H)$.

Remark

Influence of A-polynomials $\rightsquigarrow$ need (large) adapted bases to obtain similar result for relative Janet bases. Further completions necessary.

Free Resolution by Minimal Pommaret Bases

Theorem

$\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{P}$ homogeneous, G minimal Pommaret basis of $\mathcal{I}$, H minimal Pommaret basis of $\mathcal{J}$ relative to $\mathcal{I}$. For each $h \in H$ take:

1. *Syzygies $\mathbf{S}_{h,x_i}$ ($x_i \notin P_{\mathrm{lm}(\mathcal{I})}(\mathrm{lm}(h))$) induced by non-multiplicative prolongations*
2. *Syzygies $\mathbf{S}_{h,\mathbf{x}^\mu}$, where $\mathbf{x}^\mu \in (\mathrm{lm}(\mathcal{I}) : \mathrm{lm}(h)) \cap \mathbb{K}[P_{\mathrm{lm}(\mathcal{I})}(h)]$ induced by multiplicatively annihilating $\mathrm{lm}(h)$ mod $\mathrm{lm}(\mathcal{I})$*

Choose suitable enumeration of H and module monomial ordering on $(\mathcal{P}/\mathcal{I})^{|H|}$. Then $H_1 = \{\mathbf{S}_{h,x_i} \mid h \in H\} \cup \{\mathbf{S}_{h,\mathbf{x}^\mu} \mid h \in H\}$ is a minimal Pommaret basis of $\mathrm{Syz}^1_{\mathcal{P}/\mathcal{I}}(H)$.

Iteration of this construction $\rightsquigarrow$ $\mathcal{P}/\mathcal{I}$-free resolution

$$\cdots \to (\mathcal{P}/\mathcal{I})^{|H_1|} \to (\mathcal{P}/\mathcal{I})^{|H|} \to \mathcal{J}/\mathcal{I} \to 0$$

Different Concepts of Minimality

- Resolution obtained is in general not minimal (differentials may contain constants)
- Generally: minimal free resolution smaller than resolution by reduced Gröbner bases smaller than resolution by minimal Pommaret bases
- But: Pommaret resolution highly structured, combinatorial decompositions of each syzygy module $\mathrm{Syz}^k_{\mathcal{P}/\mathcal{I}}(H)$

An Example

Example

$\mathcal{P} = \mathbb{K}[x, y, z]$, $\mathcal{I} = \langle z^3 \rangle$, $\mathcal{J} = \langle xyz, y^2z, yz^2, \mathcal{I} \rangle$.

$$\cdots \to (\mathcal{P}/\mathcal{I})^4 \overset{\varphi_4}{\to} (\mathcal{P}/\mathcal{I})^4 \overset{\varphi_3}{\to} (\mathcal{P}/\mathcal{I})^4 \overset{\varphi_2}{\to} (\mathcal{P}/\mathcal{I})^4 \overset{\varphi_1}{\to} (\mathcal{P}/\mathcal{I})^3 \overset{\varphi_0}{\to} \mathcal{J}/\mathcal{I} \to 0$$

Mappings φ_k given by matrices D_k, with:

$$D_0 = \begin{pmatrix} xyz & y^2z & yz^2 \end{pmatrix}, \qquad D_1 = \begin{pmatrix} y & z & 0 & 0 \\ -x & 0 & z & 0 \\ 0 & -x & -y & z \end{pmatrix},$$

$$D_2 = \begin{pmatrix} z & 0 & 0 & 0 \\ -y & z^2 & 0 & 0 \\ x & 0 & z^2 & 0 \\ 0 & xz & yz & z^2 \end{pmatrix}, \qquad D_3 = \begin{pmatrix} z^2 & 0 & 0 & 0 \\ y & z & 0 & 0 \\ -x & 0 & z & 0 \\ 0 & -x & -y & z \end{pmatrix},$$

and $D_4 = D_2$.

Future Work/Applications

- Find conditions on $\mathcal{I}, \mathcal{J}$ under which the Pommaret resolution is minimal: relative version of concept *componentwise linear ideal*
- Study growth of degrees of entries of differentials
- Relate homological properties of $\mathcal{I}$ over $\mathcal{P}$ to those of, e.g., $\mathbb{K}$ over $\mathcal{P}/\mathcal{I}$
- Key point: impact of A-syzygies